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Fixed Point Results in G-Metric Spaces through Rational Contractive Conditions

Rajesh Shrivastava*, Manish Sharma** and Ramakant Bhardwaj***

**Professor, Department of Mathematics, Institute for Excellence in Higher Education, Bhopal, (MP) INDIA*
***Research Scholar, Department of Mathematics, Truba Institute of Engg. & Information Technology Bhopal, (MP) INDIA*
**** Professor, Department of Mathematics, TIT group of Institutes (TIT -Excellence), Bhopal, (MP) INDIA*

(Corresponding author: Manish Sharma)
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ABSTRACT: In this presented article we give some fixed point theorems in G-metric spaces through symmetric rational contractive conditions. Our theorems are generalization of various known results.

Keywords: G-metric space, fixed point, Continuous mapping, self mapping.
Mathematical Subject Classification: 54H25, 45J10.

I. INTRODUCTION

Analysis is a one of the most interesting research field of mathematics because of its wide area in applications. Metric spaces are playing an important role in mathematics and the applied sciences. Fixed point in metric space is a fundamental tool to solve various problems in applied sciences. Generalization of metric spaces is also interesting and attraction of various mathematicians. In 2003, Mustafa and Sims [1] introduced a more appropriate and robust notion of a generalized metric space known as G-metric space and prove a fixed point theorem in this space. After the notion of G-metric spaces, there are several interesting results are published by various mathematicians [2-7].

Our aim of this article is to present a fixed point theorem in G-metric space satisfying a new rational contractive condition. Our presented results are a generalization of various known results.

This article is divided into four sections. First section of this article contain brief introduction of our presented results. Second section of this article is Preliminaries in which contains some known definitions and propositions which are related to prove of our results. Third section of this article is Main results in which we give and prove our main results. Fourth section is Conclusion part of this article.

II. PRELIMINARIES

In this section we give some known definitions and results which are helpful to prove of our main theorem.

Definition 2.1: Let X be a nonempty set and let $G: X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$
(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (Symmetry in all three variables)
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (Rectangle inequality)

Then the function G is called a generalized metric or G-metric on X and (x, G) is called a G-metric space.

Definition 2.2: Let (x, G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to the limit of the sequence $\{x_n\}$, if $\lim_{n\to\infty} G(x, x_n, x_m) = 0$. Then $\{x_n\}$ is G-convergent to X.

Proposition 2.3: Let (X, G) be a G-metric space, then for any $x, y, z, a \in X$, it follows that

(i) if $G(x, y, z) = 0$ then $x = y = z$
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$
(iii) $G(x, y, y) \leq 2G(y, x, x)$
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$
(v) $G(x, y, z) \leq 2/3\ (G(x, y, a) + G(x, a, z) + G(a, y, z))$
(v) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$).

Proposition 2.4: Let (X, G) be a G-metric space, then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$. The following are equivalent

(i) $\{x_n\}$ is G-convergent to x.
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.
Definition 2.5: Let (X, G) be a G-metric space, then the sequence $\{x_n\}$ is said to be G-Cauchy if for every $\varepsilon > 0$ there exists a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$ i.e. $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.
Definition 2.6: A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).
Proposition 2.7: Let $(X, G), (X', G')$ be G-metric spaces, then a function $f: X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x i.e. whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

III. MAIN RESULTS

Our main results of this article are as follows.

Theorem 3.1 : Let (X, G) be a complete G-Metric space and T be a self –map on X satisfying,

$$G(Tx, Ty, Tz) \leq \alpha \frac{G(x,Ty,Ty)+G(x,Tz,Tz)}{2} + \beta \frac{G(x,Ty,Ty)[G(x,Ty,Ty)+G(x,Tz,Tz)+G(y,Tx,Tx)+G(z,Tx,Tx)]}{2[G(x,Ty,Ty)+G(y,Tx,Tx)]} \tag{3.1.1}$$

For all $x, y, z \in X$, where $0 \leq (\alpha + \beta) < \frac{1}{2}$. Then T has a unique fixed point u and T is G-continuous at u.

Proof : Let $x_0 \in X$ and defined the sequence $\{x_n\}$ by

$$Tx_0 = x_1, Tx_1 = x_2, Tx_2 = x_3, \dots \dots \dots \dots Tx_n = x_{n+1}$$

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

$$\leq \alpha \frac{G(x_{n-1},Tx_n,Tx_n)+G(x_{n-1},Tx_n,Tx_n)}{2}$$

$$+\beta \frac{G(x_{n-1},Tx_n,Tx_n)[G(x_{n-1},Tx_n,Tx_n)+G(x_{n-1},Tx_n,Tx_n)+G(x_n,Tx_{n-1},Tx_{n-1})+G(x_n,Tx_{n-1},Tx_{n-1})]}{2[G(x_{n-1},Tx_n,Tx_n)+G(x_n,Tx_{n-1},Tx_{n-1})]}$$

$$\leq \alpha \frac{G(x_{n-1},x_{n+1},x_{n+1})+G(x_{n-1},x_{n+1},x_{n+1})}{2}$$

$$+\beta \frac{G(x_{n-1},x_{n+1},x_{n+1})[G(x_{n-1},x_{n+1},x_{n+1})+G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_n,x_n)+G(x_n,x_n,x_n)]}{2[G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_n,x_n)]}$$

$$\leq \alpha G(x_{n-1}, x_{n+1}, x_{n+1}) + \beta G(x_{n-1}, x_{n+1}, x_{n+1})$$

$$(1 - \alpha - \beta)G(x_n, x_{n+1}, x_{n+1}) \leq (\alpha + \beta)G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{(\alpha+\beta)}{(1-\alpha-\beta)} G(x_{n-1}, x_n, x_n)$$

Let $K = \frac{(\alpha+\beta)}{(1-\alpha-\beta)}$

$$G(x_n, x_{n+1}, x_{n+1}) \leq KG(x_{n-1}, x_n, x_n) \tag{3.1.2}$$

On further decomposing we can write

$$G(x_{n-1}, x_n, x_n) \leq KG(x_{n-2}, x_{n-1}, x_{n-1}) \tag{3.1.3}$$

By combination of (3.1.2) and (3.1.3) we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq K^2 G(x_{n-2}, x_{n-1}, x_{n-1})$$

On continuing this process n times

$$G(x_n, x_{n+1}, x_{n+1}) \leq K^n G(x_0, x_1, x_1)$$

Then for all $n, m \in N$, $n < m$ we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m)$$

$$\leq (K^n + K^{n+1} + \cdots \ldots + K^{m-1}) G(x_0, x_1, x_1)$$

$$\leq \frac{K^n}{1-k} G(x_0, x_1, x_1)$$

Therefore $\{x_n\}$ is G-Cauchy sequence, hence G-convergent, since X is G-complete metric space so $\{x_n\}$ is G-converges to u.
Form (3.1.1) we have

$$G(x_n, Tu, Tu) = G(Tx_{n-1}, Tu, Tu)$$

$$\leq \alpha \frac{G(x_{n-1}, Tu, Tu) + G(x_{n-1}, Tu, Tu)}{2}$$

$$+\beta \frac{G(x_{n-1}, Tu, Tu)[G(x_{n-1}, Tu, Tu) + G(x_{n-1}, Tu, Tu) + G(u, Tx_{n-1}, Tx_{n-1}) + G(u, Tx_{n-1}, Tx_{n-1})]}{2[G(x_{n-1}, Tu, Tu) + G(u, Tx_{n-1}, Tx_{n-1})]}$$

$$\leq \alpha \frac{G(x_{n-1}, Tu, Tu) + G(x_{n-1}, Tu, Tu)}{2} + \beta \frac{G(x_{n-1}, Tu, Tu)[G(x_{n-1}, Tu, Tu) + G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n) + G(u, x_n, x_n)]}{2[G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n)]}$$

Taking the limit of both sides of above as $n \to \infty$ yields

$$G(u, Tu, Tu) \leq (\alpha + \beta) G(u, Tu, Tu)$$

This contradiction implies that $u = Tu$.
To prove uniqueness, suppose that u and v are two fixed point for T. Then

$$G(u, v, v) = G(Tu, Tv, Tv)$$

$$\leq \alpha \frac{G(u, Tv, Tv) + G(u, Tv, Tv)}{2} + \beta \frac{G(u, Tv, Tv)[G(u, Tv, Tv) + G(u, Tv, Tv) + G(v, Tu, Tu) + G(v, Tu, Tu)]}{2[G(u, Tv, Tv) + G(v, Tu, Tu)]}$$

$$G(u, v, v) \leq \alpha G(u, v, v) + \beta G(u, v, v)$$

$$G(u, v, v) \leq (\alpha + \beta) G(u, v, v)$$

$$\Rightarrow G(u, v, v) = 0$$

$$\Rightarrow u = v$$

To show that T is G-continuous at u.
Let $\{y_n\}$ be a sequences converges to u in (X, G) then we can deduce that

$$G(u, Ty_n, Ty_n) = G(Tu, Ty_n, Ty_n)$$

$$\leq \alpha \frac{G(u, Ty_n, Ty_n) + G(u, Ty_n, Ty_n)}{2}$$

$$+\beta \frac{G(u, Ty_n, Ty_n)[G(u, Ty_n, Ty_n) + G(u, Ty_n, Ty_n) + G(y_n, Tu, Tu) + G(y_n, Tu, Tu)]}{2[G(u, Ty_n, Ty_n) + G(y_n, Tu, Tu)]}$$

$$G(u, Ty_n, Ty_n) \leq (\alpha + \beta) G(u, Ty_n, Ty_n)$$

$$[1 - (\alpha + \beta)] G(u, Ty_n, Ty_n) \leq 0$$

$$G(u, Ty_n, Ty_n) \leq 0$$

Taking the limit as $n \to \infty$ from which we see that $G(u, Ty_n, Ty_n) \to 0$ and so, by proposition (2.4) we have that the sequence Ty_n is G – convergent to $Tu = u$ therefore proposition (2.7) implies that T is G-continuous at u.

Theorem 3.2: Let (X, G) be complete G-metric space and let $T: X \to X$ be a mapping satisfying the condition

$$G(Tx, Ty, Tz) \le \alpha \min\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), G(x, y, z)\}$$

$$+\beta\left[\frac{G(x,Tx,Tx)+G(y,Tx,Tx)+G(x,Ty,Ty)}{1+G(x,Tx,Tx)\,G(y,Tx,Tx)\,G(x,Ty,Ty)}\right] \tag{3.2.1}$$

For all $x, y, z \in X$ Where $\alpha, \beta, \ge 0$ and $\alpha + 3\beta < 1$ Then T has unique fixed point u and T is G-continuous at u.
Proof: Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n x_0$ then by condition

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

$$\le \alpha \min\left\{\begin{matrix} G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\ G(x_n, Tx_n, Tx_n), G(x_{n-1}, x_n, x_n) \end{matrix}\right\}$$

$$+\beta\left[\frac{G(x_{n-1},Tx_{n-1},Tx_{n-1})+G(x_n,Tx_{n-1},Tx_{n-1})+G(x_{n-1},Tx_n,Tx_n)}{1+G(x_{n-1},Tx_{n-1},Tx_{n-1})\,G(x_n,Tx_{n-1},Tx_{n-1})\,G(x_{n-1},Tx_n,Tx_n)}\right]$$

$$\le \alpha \min\left\{\begin{matrix} G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n) \end{matrix}\right\}$$

$$+\beta\left[\frac{G(x_{n-1},x_n,x_n)+G(x_n,x_n,x_n)+G(x_{n-1},x_{n+1},x_{n+1})}{1+G(x_{n-1},x_n,x_n)\,G(x_n,x_n,x_n)\,G(x_{n-1},x_{n+1},x_{n+1})}\right]$$

$$\le \alpha \min\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}$$

$$+\beta[G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})] \tag{3.2.2}$$

Here two cases are arise

Case – I : If $\min\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_{n-1}, x_n, x_n)$

Then condition (3.2.2) reduces to

$$G(x_n, x_{n+1}, x_{n+1}) \le \alpha G(x_{n-1}, x_n, x_n) + \beta[G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})]$$

$$(1 - \beta)G(x_n, x_{n+1}, x_{n+1}) \le (\alpha + 2\beta)G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{(\alpha+2\beta)}{(1-\beta)} G(x_{n-1}, x_n, x_n)$$

Let $K = \frac{(\alpha+2\beta)}{(1-\beta)}$

$$G(x_n, x_{n+1}, x_{n+1}) \le KG(x_{n-1}, x_n, x_n)$$

On continuing this process n times

$$G(x_n, x_{n+1}, x_{n+1}) \le K^n G(x_0, x_1, x_1)$$

Case – II : If $\min\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} = G(x_n, x_{n+1}, x_{n+1})$
Then condition (3.2.2) reduces to

$$G(x_n, x_{n+1}, x_{n+1}) \le \alpha G(x_n, x_{n+1}, x_{n+1}) + \beta[G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n+1}, x_{n+1})]$$

$$(1 - \alpha - \beta)G(x_n, x_{n+1}, x_{n+1}) \le 2\beta\, G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{2\beta}{(1-\alpha-\beta)} G(x_{n-1}, x_n, x_n)$$

Let $K = \frac{2\beta}{(1-\alpha-\beta)}$

$$G(x_n, x_{n+1}, x_{n+1}) \le KG(x_{n-1}, x_n, x_n)$$

On continuing this process n times

$$G(x_n, x_{n+1}, x_{n+1}) \leq K^n G(x_0, x_1, x_1)$$

Then for all $n, m \in N$, $n < m$ we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m)$$

$$\leq (K^n + K^{n+1} + \cdots \ldots \ldots \ldots + K^{m-1}) G(x_0, x_1, x_1)$$

$$\leq \frac{K^n}{1-K} G(x_0, x_1, x_1)$$

Therefore $\{x_n\}$ is G- Cauchy sequence, Hence G-convergent, since X is
G-complete metric space so X is G-converges to u.

Form (3.2.1) we have

$$G(u, Tu, Tu) = G(x_n, Tu, Tu) = G(Tx_{n-1}, Tu, Tu)$$

$$\leq \alpha \min\{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(u, Tu, Tu), G(u, Tu, Tu), G(x_{n-1}, u, u)\}$$

$$+\beta \left[\frac{G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(u, Tx_{n-1}, Tx_{n-1}) + G(x_{n-1}, Tu, Tu)}{1 + G(x_{n-1}, Tx_{n-1}, Tx_{n-1})\, G(u, Tx_{n-1}, Tx_{n-1})\, G(x_{n-1}, Tu, Tu)}\right]$$

$$\leq \alpha \min\{G(x_{n-1}, x_n, x_n), G(u, Tu, Tu), G(u, Tu, Tu), G(x_{n-1}, u, u)\}$$

$$+\beta \left[\frac{G(x_{n-1}, x_n, x_n) + G(u, x_n, x_n) + G(x_{n-1}, Tu, Tu)}{1 + G(x_{n-1}, x_n, x_n)\, G(u, x_n, x_n)\, G(x_{n-1}, Tu, Tu)}\right]$$

Taking the limit as taking the limit as $n \to \infty$

$$G(u, Tu, Tu) \leq \beta G(u, Tu, Tu)$$

Since $\beta < 1$.

Which implies that $G(u, Tu, Tu) = 0$
And hence u = Tu.
To prove uniqueness suppose that u and v are two fixed point for T.

Then

$$G(u, v, v) = G(Tu, Tv, Tv)$$

$$\leq \alpha \min\{G(u, Tu, Tu), G(v, Tv, Tv), G(v, Tv, Tv), G(u, v, v)\}$$

$$+\beta \left[\frac{G(u, Tu, Tu) + G(v, Tu, Tu) + G(u, Tv, Tv)}{1 + G(u, Tu, Tu)\, G(v, Tu, Tu)\, G(u, Tv, Tv)}\right] \leq \beta G(v, u, u)$$

$G(u, v, v) \leq 2\beta G(u, v, v)$ a contradiction. Therefore, $G(u, v, v) = 0$.

Hence u = v
Let $\{y_n\} \subset X$ be any sequence with limit u

Using (3.2.1)

$$G(u, Ty_n, Ty_n) = G(Tu, Ty_n, Ty_n)$$

$$\leq \alpha \min\{G(u, Tu, Tu), G(y_n, Ty_n, Ty_n), G(y_n, Ty_n, Ty_n), G(u, y_n, y_n)\}$$

$$+\beta \left[\frac{G(u, Tu, Tu) + G(y_n, Tu, Tu) + G(u, Ty_n, Ty_n)}{1 + G(u, Tu, Tu) G(y_n, Tu, Tu) G(u, Ty_n, Ty_n)}\right]$$

$$\leq \beta [G(y_n, u, u) + G(u, Ty_n, Ty_n)]$$

$$G(u, Ty_n, Ty_n) \leq \frac{\beta}{1-\beta} G(y_n, u, u)$$

Taking the limit as $n \to \infty$ from which we see that $G(u, Ty_n, Ty_n) \to 0$ and so, by proposition (2.4) we have that the sequence Ty_n is G – convergent to $Tu = u$ therefore proposition (2.7) implies that T is G-continuous at u.

Theorem 3.3 : Let (X, G) be a complete G- Metric space and let T be a self-map on X satisfying,

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + \beta[G(x, Ty, Ty) + G(x, Tz, Tz)]$$

$$+\gamma \frac{G(x,y,z)[1+G(x,Ty,Ty)]}{1+G(x,Tz,Tz)} + \delta\left[\frac{G(x,Ty,Ty).G(y,Ty,Ty)}{G(z,Ty,Ty)}\right] \tag{3.3.1}$$

For all x, y, z in X with $G(z, Ty, Ty) \neq 0$ Where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + 4\beta + \gamma + 2\delta < 1$

Proof : Let x_0 □ X and define the sequence $\{x_n\}$ by

$$Tx_0 = x_1, Tx_1 = x_2, \ldots\ldots\ldots, Tx_n = x_{n+1}$$

Here we may assume that $x_n \neq x_{n+1}$

Consider,

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

$$\leq \alpha G(x_{n-1}, x_n, x_n) + \beta[G(x_{n-1}, Tx_n, Tx_n) + G(x_{n-1}, Tx_n, Tx_n)]$$

$$+\gamma \frac{G(x_{n-1},x_n,x_n)[1+G(x_{n-1},Tx_n,Tx_n)]}{1+G(x_{n-1},Tx_n,Tx_n)} + \delta\left[\frac{G(x_{n-1},Tx_n,Tx_n).G(x_n,Tx_n,Tx_n)}{G(x_n,Tx_n,Tx_n)}\right]$$

$$\leq \alpha G(x_{n-1}, x_n, x_n) + \beta[G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1})]$$

$$+\gamma \frac{G(x_{n-1},x_n,x_n)[1+G(x_{n-1},x_{n+1},x_{n+1})]}{1+G(x_{n-1},x_{n+1},x_{n+1})} + \delta\left[\frac{G(x_{n-1},x_{n+1},x_{n+1}).G(x_n,x_{n+1},x_{n+1})}{G(x_n,x_{n+1},x_{n+1})}\right]$$

$$\leq \alpha G(x_{n-1}, x_n, x_n) + 2\beta G(x_{n-1}, x_{n+1}, x_{n+1}) + \gamma G(x_{n-1}, x_n, x_n) + \delta G(x_{n-1}, x_{n+1}, x_{n+1})$$

$$\leq \alpha G(x_{n-1}, x_n, x_n) + 2\beta[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})]$$

$$+\gamma G(x_{n-1}, x_n, x_n) + \delta[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})]$$

$$(1 - 2\beta - \delta)G(x_n, x_{n+1}, x_{n+1}) \leq (\alpha + 2\beta + \gamma + \delta)G(x_{n-1}, x_n, x_n)$$

$$\leq \frac{\alpha+2\beta+\gamma+\delta}{1-2\beta-\delta} G(x_{n-1}, x_n, x_n)$$

Let $\frac{\alpha+2\beta+\gamma+\delta}{1-2\beta-\delta} = K$

□ $G(x_n, x_{n+1}, x_{n+1}) \leq KG(x_{n-1}, x_n, x_n)$

On continuing this process n times

$$G(x_n, x_{n+1}, x_{n+1}) \leq K^n G(x_0, x_1, x_1)$$

Then for all $m, n \in N, n < m$ we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + G(x_{m-1}, x_m, x_m)$$

$$\leq (K^n + K^{n+1} + \cdots\ldots\ldots\ldots + K^{m-1})G(x_0, x_1, x_1)$$

$$\leq \frac{K^n}{1-K} G(x_0, x_1, x_1)$$

Therefore $\{x_n\}$ is G- Cauchy sequence, Hence G-convergent, since X is G-complete metric space so X is G-converges to u.

Form (3.3.1) we have

$$G(x_n, Tu, Tu) = G(Tx_{n-1}, Tu, Tu)$$

$$\leq \alpha G(x_{n-1}, u, u) + \beta[G(x_{n-1}, Tu, Tu) + G(x_{n-1}, Tu, Tu)]$$

$$+\gamma \frac{G(x_{n-1}, u, u)[1 + G(x_{n-1}, Tu, Tu)]}{1 + G(x_{n-1}, Tu, Tu)} + \delta\left[\frac{G(x_{n-1}, Tu, Tu).G(u, Tu, Tu)}{G(u, Tu, Tu)}\right]$$

Taking the limit as taking the limit as $n \to \infty$

$$G(u, Tu, Tu) \leq (2\beta + \delta)G(u, Tu, Tu)$$

Since $(2\beta + \delta) < 1$

$$G(u, Tu, Tu) = 0$$

And hence $u = Tu$

To prove uniqueness suppose that u and v are two fixed point for T.

Then

$$G(u, v, v) = G(Tu, Tv, Tv)$$

$$\leq \alpha G(u, v, v) + \beta[G(u, Tv, Tv) + G(u, Tv, Tv)]$$

$$+\gamma \frac{G(u, v, v)[1 + G(u, Tv, Tv)]}{1 + G(u, Tv, Tv)} + \delta\left[\frac{G(u, Tv, Tv).G(v, Tv, Tv)}{G(v, Tv, Tv)}\right]$$

$$\square\ G(u, v, v) \leq (\alpha + 2\beta + \gamma + \delta)G(u, v, v)$$

Since $(\alpha + 2\beta + \gamma + \delta) < 1$

$$\square\ G(u, v, v) = 0$$

Let $\{y_n\}$ be a sequences converges to u in (X, G) then we can deduce that

$$G(u, Ty_n, Ty_n) = G(Tu, Ty_n, Ty_n)$$

$$\leq \alpha G(u, y_n, y_n) + \beta[G(u, Ty_n, Ty_n) + G(u, Ty_n, Ty_n)]$$

$$+\gamma \frac{G(u, y_n, y_n)[1 + G(u, Ty_n, Ty_n)]}{1 + G(u, Ty_n, Ty_n)} + \delta\left[\frac{G(u, Ty_n, Ty_n).G(y_n, Ty_n, Ty_n)}{G(y_n, Ty_n, Ty_n)}\right]$$

$$\Longrightarrow G(u, Ty_n, Ty_n) \leq \frac{\alpha + \gamma}{[1 - (2\beta + \delta)]} G(u, y_n, y_n)$$

Taking the limit as $n \to \infty$ from which we see that $G(u, Ty_n, Ty_n) \to 0$ and so, by proposition (2.4) we have that the sequence Ty_n is G – convergent to $Tu = u$ therefore proposition (2.7) implies that T is G- continuous at u.

CONCLUSION

In the above proved theorems we use a continuous self mapping in G-metric space and prove fixed point. We also use the concept of symmetric rational contraction mapping to prove of the theorem.

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